Then $AM + CB = AM + PM = l$. Likewise, $AM' + C'B' = l$. There will be two solutions, one solution, or no solution according as PQ cuts the circle, is tangent to the circle, or lies outside of the circle. In other words the problem is possible only when $2r < l < (1 + \sqrt{5})r$.

Also solved analytically, by H. C. FEEMSTER.

413. Proposed by DAVID R. KELLEY, New York City.

To construct a triangle, given the base, the vertical angle, and the ratio of the altitude from the vertex on the base to the difference of the other two sides.

SOLUTION BY THE PROPOSER.

Let ABC be the required triangle. Then since side AB and $\angle C$ are known, we can construct the circumcircle of the triangle. Let the bisectors of the interior angle C and the exterior angle C meet the circle in D and E, respectively. Then D and E are fixed points, and ED is the diameter of the circle, and AE is fixed. Draw EG perpendicular to BC produced. Then $CG = \frac{1}{2}(AC - BC)$. Hence, if CP be the altitude from C on AB, CG/CP is also known, since by hypothesis $(AC - CB)/CP$ is the given ratio. Again, since in the right triangle ECG, $\angle GCE$ is known, therefore GC/EC is known, and therefore, by above, EC/CP is known. Now let EC meet AB in F. Then triangles DEC and CFP are similar. Hence, $EC/CP = DE/CF$. But EC/CP is given. Hence DE/CF is also known, and therefore, since DE is the diameter of the circumcircle of triangle ABC, CF can be found. Also $EC \cdot EF = \overline{EA}^2$. Hence, since CF and EA are known, EC can be constructed and hence point C can be found. See Fig. 2.

Also solved by H. C. FEEMSTER, C. N. SCHMALL, and A. M. HARDING.

414. Proposed by H. C. FEEMSTER, York, Neb.

To construct the cyclic quadrilateral, having given its four sides.

SOLUTION BY THE PROPOSER.

Construction.—Let the lengths of the given sides be a, b, c, d. Draw $AD = d$, and produce it to E so that $DE : c = a : b$. Now describe a circle which is the locus of a point P such that $EP : PA = c : b$ (circle of Apollonius). Describe also a circle with the center D and radius c. Let C be one point of intersection of these circles. Then on AC as a base construct a triangle ABC similar to EDC. Then $ABCD$ is the quadrilateral required. See Fig. 3.

Proof.—By construction $AD = d$, $DC = c$. Now since the angles ABC and EDC are equal, therefore $\angle ABC$ and $\angle ADC$ are supplementary and the figure $ABCD$ is cyclic. It now remains to prove that $AB = a$, and $BC = b$.

By construction, $ED : c = a : b$, or $ED : DC = a : b$. Also by construction $EC : CA = c : b$, or $EC : DC = CA : b$. Hence

$$ED : DC : EC = a : b : CA. \tag{1}$$

But by similar triangles, we have

$$ED : DC : EC = AB : BC : CA. \tag{2}$$

Then by (1) and (2), $AB = a$, $BC = b$.

There is evidently but one solution; for if the other point of intersection of the circles be taken instead of C, the resulting quadrilateral will be the same in size but drawn on the opposite side of AD since the intersections are symmetrical with regard to AD.

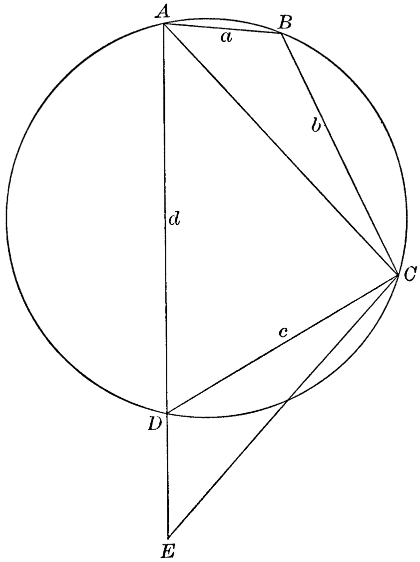

Fig. 3.

Note.—In this construction it is assumed that the order of the sides a, b, c, d, is given. If this order is not given it may be shown that three cyclic quadrilaterals can be drawn having these sides.

MECHANICS.

265. Proposed by A. H. HOLMES, Brunsw'ck, Maine.

A gun is mounted in a fort at height h above the sea, and a similar gun is mounted on a ship. Show that there is a region of area $4\pi rh$ within which the ship is within range of the fort while the fort is out of range of the ship, r being the maximum range of either gun on a horizontal plane through it.

SOLUTION BY A. M. HARDING, University of Arkansas.

If a projectile is fired at an angle α its range R on a level plane at a distance h below the point of projection is given by the equation

$$2v^2(h + R\tan\alpha) = gR^2\sec^2\alpha.$$

That is

$$R = \frac{v}{g}\cos\alpha\,[v\sin\alpha + \sqrt{(v^2 + 2gh) - v^2\cos^2\alpha}].$$

From the value of $dR/d\alpha$, set equal to zero, we have as the condition for a maximum range,

$$\sin^2\alpha = \frac{v^2}{2(v^2 + gh)} \qquad \text{and} \qquad \cos^2\alpha = \frac{v^2 + 2gh}{2(v^2 + gh)}.$$